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AUTHOR(S):

Nakane, Shizuo

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Postcritical sets and saddle basic sets for Axiom A polynomial skew products on \mathbb{C}^2

東京工芸大学 中根静男 (Shizuo Nakane)

Tokyo Polytechnic University

1 Introduction

We consider Axiom A regular polynomial skew products on \mathbb{C}^2 . It is of the form : $f(z, w) = (p(z), q(z, w))$, where $p(z) = z^d + \dots$ and $q_z(w) = q(z, w) = w^d + \dots$ are polynomials of degree $d \geq 2$. Then its k -th iterate is expressed by :

$$f^{\circ k}(z, w) = (p^{\circ k}(z), q_{p^{k-1}(z)} \circ \dots \circ q_z(w)) =: (p^{\circ k}(z), Q_z^{\circ k}(w)).$$

Hence it preserves the family of *fibers* $\{z\} \times \mathbb{C}$ and this makes it possible to study its dynamics more precisely. Let K be the set of points with bounded orbits and set $K_z := \{w \in \mathbb{C}; (z, w) \in K\}$ and $K_{J_p} := K \cap (J_p \times \mathbb{C})$. The *fiber Julia set* J_z is the boundary of K_z .

Let Ω be the set of *non-wandering points* for f . Then f is said to be *Axiom A* if Ω is compact, hyperbolic and periodic points are dense in Ω . For polynomial skew products, Jonsson [J2] has shown that f is Axiom A if and only if the following three conditions are satisfied :

- (a) p is hyperbolic,
- (b) f is vertically expanding over J_p ,
- (c) f is vertically expanding over $A_p := \{\text{attracting periodic points of } p\}$.

Here f is *vertically expanding over* $Z \subset \mathbb{C}$ with $p(Z) \subset Z$ if there exist $\lambda > 1$ and $C > 0$ such that $|(Q_z^{\circ k})'(w)| \geq C\lambda^k$ holds for any $z \in Z, w \in J_z$ and $k \geq 0$.

We are interested in the dynamics of f on $J_p \times \mathbb{C}$ because the dynamics outside $J_p \times \mathbb{C}$ is fairly simple. Consider the critical set

$$C_{J_p} = \{(z, w) \in J_p \times \mathbb{C}; q'_z(w) = 0\}$$

over the *base Julia set* J_p . Let μ be the ergodic measure of maximal entropy for f (see Fornaess and Sibony [FS1]). Its support J_2 is called the *second Julia set* of f . Let $D_{J_p} := \overline{\cup_{n \geq 1} f^{\circ n}(C_{J_p})}$ be the *postcritical set* of C_{J_p} . Jonsson [J2] has shown that

- (d) $J_2 = \overline{\cup_{z \in J_p} \{z\} \times J_z}$,
- (e) the condition (b) $\iff D_{J_p} \cap J_2 = \emptyset$,
- (f) J_2 is the closure of the set of repelling periodic points of f .

By the Birkhoff ergodic theorem, μ -a.e. x has a dense orbit in J_2 . Especially, $J_2 = \text{supp } \mu$ is transitive. Hence J_2 coincides with the *basic set* of unstable dimension two. See also [FS2].

For any subset X in \mathbb{C}^2 , its accumulation set is defined by

$$A(X) = \bigcap_{N \geq 0} \overline{\bigcup_{n \geq N} f^{\circ n}(X)}.$$

DeMarco & Hruska [DH1] defined the *pointwise* and *component-wise* accumulation sets of C_{J_p} respectively by

$$A_{pt}(C_{J_p}) = \overline{\bigcup_{x \in C_{J_p}} A(x)} \quad \text{and} \quad A_{cc}(C_{J_p}) = \overline{\bigcup_{C \in \mathcal{C}(C_{J_p})} A(C)},$$

where $\mathcal{C}(C_{J_p})$ denotes the collection of connected components of C_{J_p} . It follows from the definition that

$$A_{pt}(C_{J_p}) \subset A_{cc}(C_{J_p}) \subset A(C_{J_p}).$$

It also follows that $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p})$ if J_p is a Cantor set and $A_{cc}(C_{J_p}) = A(C_{J_p})$ if J_p is connected.

Let Λ be the closure of the set of saddle periodic points in $J_p \times \mathbb{C}$. It decomposes into a disjoint union of *saddle basic sets* : $\Lambda = \sqcup_{i=1}^m \Lambda_i$. Put $\Lambda_z = \{w \in \mathbb{C}; (z, w) \in \Lambda\}$. The *stable* and *unstable sets* of Λ , the *local stable* and *local unstable manifolds* of $x \in \Lambda$ and $\hat{x} \in \hat{\Lambda}$ are respectively defined by

$$\begin{aligned} W^s(\Lambda) &= \{y \in \mathbb{C}^2; f^{\circ k}(y) \rightarrow \Lambda\}, \\ W^u(\Lambda) &= \{y \in \mathbb{C}^2; \exists \text{ backward orbit } \hat{y} = (y_{-k}) \text{ tending to } \Lambda\}, \\ W_\delta^s(x) &= \{y \in \mathbb{C}^2; \|f^{\circ k}(y) - f^{\circ k}(x)\| < \delta, \forall k \geq 0\}, \\ W_\delta^s(\hat{x}) &= \{y \in \mathbb{C}^2; \exists \hat{y} \text{ s.t. } \|y_{-k} - x_{-k}\| < \delta, \forall k \geq 0\}. \end{aligned}$$

On Λ , f is contracting in the fiber direction and

$$W_\delta^s(x) \subset \{z\} \times \mathbb{C}, \quad x = (z, w) \in \Lambda.$$

Theorem A. ([DH1])

$$A_{pt}(C_{J_p}) = \Lambda, \quad A(C_{J_p}) = W^u(\Lambda) \cap (J_p \times \mathbb{C}).$$

Theorem B. ([DH1, DH2])

$$A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \implies \forall C \in \mathcal{C}(C_{J_p}), \quad C \cap K = \emptyset \text{ or } C \subset K. \quad (1)$$

Theorem C. ([DH1, DH2])

$$A(C_{J_p}) = A_{pt}(C_{J_p}) \iff \text{the map } z \mapsto \Lambda_z \text{ is continuous in } J_p. \quad (2)$$

Under the assumption $W^u(\Lambda) \cap W^s(\Lambda) = \Lambda$,

$$A(C_{J_p}) = A_{pt}(C_{J_p}) \iff \text{the map } z \mapsto K_z \text{ is continuous in } J_p. \quad (3)$$

As for the assumption in the above theorem, we have

Lemma 1. $W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \iff W^u(\Lambda_i) \cap W^s(\Lambda_j) = \emptyset$ if $i \neq j$.

Sumi [S] gives an example of Axiom A polynomial skew product which does not satisfy the condition in the above lemma. See the last section. It is also (incorrectly) described as Example 5.10 in [DH1]. See also [DH2].

We define a relation \succ among saddle basic sets by

$$\Lambda_i \succ \Lambda_j \quad \text{if} \quad (W^u(\Lambda_i) \setminus \Lambda_i) \cap (W^s(\Lambda_j) \setminus \Lambda_j) \neq \emptyset.$$

A *cycle* is a chain of basic sets :

$$\Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots \succ \Lambda_{i_n} = \Lambda_{i_1}.$$

For Axiom A open endomorphisms, there is no trivial cycle because $W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i$ holds for any i . See [J2], Proposition A.4. Jonsson has also shown that, for Axiom A polynomial skew products on \mathbb{C}^2 , the non-wandering set Ω coincides with the *chain recurrent set* \mathcal{R} . This leads to the following lemma.

Lemma 2. ([J2], Corollary 8.14) *Axiom A polynomial skew products on \mathbb{C}^2 have no cycles.*

Set $\Lambda_0 := \emptyset$, $W^s(\Lambda_0) := (J_p \times \mathbb{C}) \setminus K$ and $C_i := C_{J_p} \cap W^s(\Lambda_i)$ ($0 \leq i \leq m$). If we consider in \mathbb{P}^2 , Λ_0 should be the superattracting fixed point $\{[0 : 1 : 0]\}$.

We will give characterizations of the equalities $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ and $A_{pt}(C_{J_p}) = A(C_{J_p})$ in terms of C_i .

Lemma 3. $C_{J_p} = \sqcup_{i=0}^m C_i$.

Note that $A(C_i) \supset A_{pt}(C_i) = \Lambda_i$ for any $i \geq 0$.

The author would like to thank Hiroki Sumi for helpful discussion on his example.

2 Results

Theorem 1.

$$A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \iff \forall C \in \mathcal{C}(C_{J_p}), 0 \leq \exists i \leq m \text{ such that } C \subset C_i. \quad (4)$$

In terms of C_i , the condition in (1) is expressed by

$$\forall C \in \mathcal{C}(C_{J_p}), \quad C \subset C_0 \text{ or } C \subset \bigcup_{i=1}^m C_i.$$

Hence, if $m = 1$, that is, Λ itself is a basic set, then the condition in (4) coincides with that in (1). In general, the condition in (4) is stronger than that in (1).

We have another characterization of $A_{pt}(C_{J_p}) = A(C_{J_p})$ in terms of C_i .

Theorem 2. *For any $i \geq 0$, we have*

$$A(C_i) = \Lambda_i \iff C_i \text{ is closed.} \quad (5)$$

Consequently we have

$$A_{pt}(C_{J_p}) = A(C_{J_p}) \iff C_i \text{ is closed for any } i \geq 0.$$

As for the condition in (3), we have

Theorem 3. *The following three conditions are equivalent to each other.*

- (a) C_0 is closed,
- (b) $A(C_{J_p}) = W^u(\Lambda) \cap W^s(\Lambda)$,
- (c) the map $z \mapsto K_z$ is continuous in J_p .

As a corollary, we get the following.

Corollary 1. $W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \iff C_i \text{ is closed for any } i \geq 1.$

As for the condition in (2), we have the following.

Theorem 4. *For each $j \geq 1$,*

$$\begin{aligned} C_j \text{ is open in } C_{J_p} &\iff W^u(\Lambda_j) \cap (J_p \times \mathbb{C}) = \Lambda_j \\ &\iff z \mapsto \Lambda_{j,z} \text{ is continuous in } J_p. \end{aligned}$$

Consequently,

$$\begin{aligned} \forall j \geq 1, C_j \text{ is open in } C_{J_p} &\iff W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda \\ &\iff z \mapsto \Lambda_z \text{ is continuous in } J_p. \end{aligned}$$

Recall that $C_0 = C_{J_p} \setminus K$ is always open in C_{J_p} .

3 Sumi's example

Sumi [S] considers the following example.

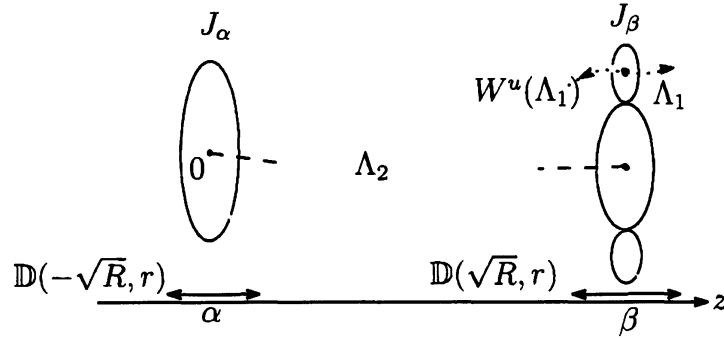
$$f(z, w) = \left((z^2 - R)^{\circ n}, w^{2^n} + \frac{z + \sqrt{R}}{2\sqrt{R}} t_{n,\epsilon}(w) \right).$$

Here $R \gg 1$, $0 < \epsilon \ll 1$, n is even and large, and

$$t_{n,\epsilon}(w) = ((w - \epsilon)^2 - 1 + \epsilon)^{\circ n} - w^{2^n}.$$

Let $\alpha < 0$ and $\beta > 0$ be the fixed points of $z^2 - R$. It satisfies the following.

- J_p is a Cantor set in $\mathbb{D}(-\sqrt{R}, r) \cup \mathbb{D}(\sqrt{R}, r)$ for some r .
- J_α is a quasicircle, while J_β is a basilica.
- $\Lambda = \Lambda_1 \sqcup \Lambda_2$, where $\Lambda_1 \subset \{\beta\} \times \mathbb{C}$ is a single point.
- $C_{J_p} \subset K$, i.e. $C_0 = \emptyset$, hence $z \mapsto K_z$ is continuous in J_p .
- $C_1 \subset \{\beta\} \times \mathbb{C}$ is a finite set.
- $C_2 = C_{J_p} \setminus C_1$ is open in C_{J_p} and $\overline{C_2} \supset C_1$.
- $W^u(\Lambda_1) \cap W^s(\Lambda_2) \setminus \Lambda \neq \emptyset$, i.e. $\Lambda_1 \succ \Lambda_2$.
- The map $z \mapsto \Lambda_{2,z}$ is continuous in J_p .
- $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p}) \neq A(C_{J_p})$.



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